ITERATED MONODROMY GROUP OF RATIONAL MAPS

A Thesis

by

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ABSTRACT

We describe the iterated monodromy groups of rational functions of the form $f_c(z) = 1 + c/z^2$, where the parameter c lies in certain components of the parameter plane which are attached to the main cardioid. Such a group is determined uniquely by a rational angle determined by the parameter.

We give a proof for iterated monodromy groups of such parameters and then give iterated monodromy groups of parameters lying in finer limbs by a so-called tuning method.

CONTRIBUTORS AND FUNDING SOURCES

Contributors

This work was supported by a thesis committee consisting of Professor Volodymyr Nekrashevych [advisor] and Rostislav Grigorchuk of the Department of Mathematics and Professor Michael Longnecker of the Department of Statistics.

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1. INTRODUCTION AND LITERATURE REVIEW

The iterated monodromy group is a self-similar group which has strong connection with dynamical systems[1]. It can be naturally defined for many dynamical systems, for example in our case, the iteration of a rational function on the Riemann sphere $\hat{\mathbb{C}}$. Recently, the theory of iterated monodromy groups has been studied a lot for its both applications to the study of dynamical systems and interesting algebraic properties.

The iterated monodromy groups originated from the problem of computing the Galois group of the extension $\Omega/\mathbb{C}(t)$, where $\Omega = \bigcup_{n\geq 1} \Omega_n$, and Ω_n is the field obtained by adjoining to the field of functions $\mathbb{C}(t)$ all solutions of the equation $F^n(x) = t$ in an algebraic closure of $\mathbb{C}(t)$, for some rational function $F(x)$. Then it was defined by Nekrashevych as an discrete group, and later applied to a solution of the Hubbard twisted rabbit problem in complex dynamics. The iterated monodromy group often contains the essential information about the dynamical system. For example, one can reconstruct the Julia set of the system from the iterated monodromy group in the hyperbolic case.

On the other hand, the iterated monodromy groups have also been studied for their growth properties. The growth of a finitely generated group is the volume growth of balls in the Cayley graph of the group. For example, the free group of a finite rank has an exponential growth rate, and a finite group has constant growth. Iterated Monodromy groups may give examples of groups of intermediate growth, similar to the first example of a finitely generated group of intermediate growth given by Grigorchuk.

The iterated monodromy groups of postcritically-finite polynomials are well studied. One class of examples are for quadratic polynomials. Their iterated monodromy groups can be computed via the kneading sequence, see [2]. Other iterated monodromy groups of polynomials can also be given in terms of automata generating them, see [3]. However, the iterated monodromy groups of rational maps other than polynomials are much less understood. So it will be nice to give a description of the iterated monodromy groups of some families of rational maps in terms of some combinatorial information.

2. BACKGROUND

2.1 Basic Definitions

The iterated monodromy group can be defined as follows. Here we consider the iterated monodromy group of a topological dynamical system.

- Let M be a path connected and locally path connected topological space, and let $p : M_1 \to M$ be a degree $d > 1$ covering map, where $M_1 \subseteq M$ is a subset of M. We call such maps $p : M_1 \to M$ *partial self-coverings* of M.

- Choose a point $t \in M$ and consider the fundamental group $\pi_1(M, t)$ acting on the rooted tree of preimages with the vertex set $T_p = \bigcup_{n \geq 0} p^{-n}(t)$, where a vertex $z \in p^{-(n+1)}(t)$ is connected by an edge with the vertex $p(z) \in p^{-n}(t)$.

- Since the action is defined by lifting the loops by p, the action of the fundamental group on the levels of the tree agrees with the tree structure, and we get an action of $\pi_1(M, t)$ on T_p by automorphisms of the rooted tree. This action is called the *iterated monodromy action*.

- The quotient of the fundamental group by the kernel of the iterated monodromy action is called the *iterated monodromy group* of p and is denoted as $IMG(p)$.

Now it is clear that the iterated monodromy group acts on a regular rooted tree by automorphisms. In order to compute them explicitly, we want to encode the regular rooted tree by finite words over some alphabet X. Note that there is a natural way to label the rooted tree. Denote the root as $\emptyset$, and for vertices which are children of a vertex with label v, we give them labels vx respectively, where $x \in X$.

More explicitly, for our preimage tree, we can encode it using lifts of paths in the following way. Let X be $[d]$ where d is the degree of the partial self-covering p. First set $\Lambda(\emptyset) = t$, determine a bijection $\Lambda : X \to p^{-1}(t)$, and a path $\ell(x)$ from t to $\Lambda(x)$ for each $x \in X$, where t is the root.

Then define $\Lambda : X^* \to T_p$ inductively by letting $\Lambda(xv)$ to be the end of the $p^{|v|}$-lift of $\ell(x)$ starting at $\Lambda(v)$.

Under this notation, we introduce the main proposition of [1] used to compute the iterated monodromy group.

Proposition 1. *Let γ be an element of the fundamental group $\pi_1(M, t)$. For $x \in X$, let γ_x be the lift of γ by p starting at $\Lambda(x)$. Let $y \in X$ be such that $\Lambda(y)$ is the end of γ_x. Then for every $v \in X^*$ we have*

$$\gamma(xv) = y(l(x)\gamma_x l^{-1}(y))(v).$$

2.2 Notation

Proposition 1 actually proves that the iterated monodromy group is self-similar. Recall the definition of self-similar group.

Definition 1. *A group G acting faithfully on the set X^* is called self-similar if for every $g \in G$ and $x \in X$ there exist $h \in G$ such that*

$$g(xw) = g(x)h(w)$$

for all $w \in X^*$. Here the element h is uniquely determined, and is called the section or restriction of g in x, denoted as $g|_x$.

Since the iterated monodromy group is a self-similar group, we can denote its elements in terms of their restrictions. For our function $f_c(z) = 1 + c/z^2$ which has degree 2, we will use the notation $a = \ll a|_0, a|_1 \gg \sigma^i$, where σ is the permutation $(01) \in S_2$, the order of σ is either 0 or 1.

Also recall how the group $\mathfrak{R}(v)$ for quadratic polynomials are defined in [2]. The group $\mathfrak{R}(v)$ for any word $v = x_1 x_2 \dots x_n \in \{0, 1\}^n$ is defined as

$$a_1 = \ll 1, a_n \gg \sigma, \qquad a_{i+1} = \begin{cases} \ll a_i, 1 \gg & \text{if } x_i = 0, \\ \ll 1, a_i \gg & \text{if } x_i = 1, \end{cases} \quad \text{when } 1 \le i < n.$$

3. ITERATED MONODROMY GROUPS OF FUNCTION $f_c(z) = 1 + c/z^2$

3.1 Dynamical Behavior of f_c

First let's consider the dynamical properties of f_c. Note that our function f_c has degree 2, and the only critical points are 0 and ∞. Thus the critical orbit goes like:

$$0 \Rightarrow \infty \Rightarrow 1 \rightarrow 1 + c \rightarrow \ldots$$

One thing which is worth notice is that by reversing $z \mapsto 1/z$, we get the quadratic polynomial again, with parameter $1/c$. So one can image the relation of their dynamical planes between quadratic polynomials and f_c. For instance, there will be copies of Mandelbrot set and Julia sets of quadratic polynomials appearing in the parameter plane and dynamical planes of f_c respectively.

Here is the parameter plane of f_c. All fractal figures in this thesis are made via software Mandel, see [4].

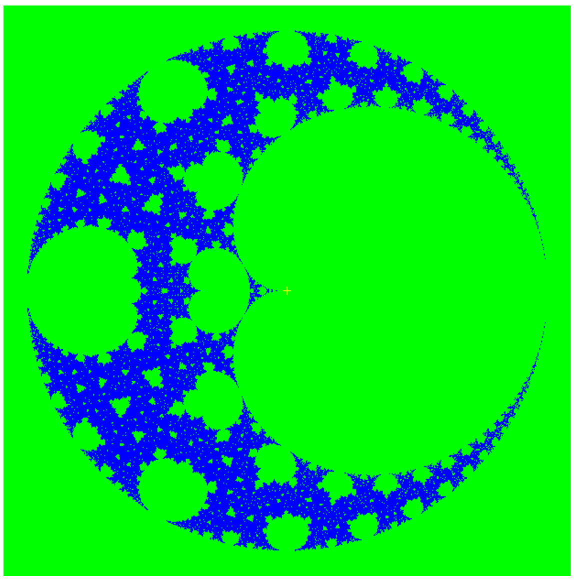

Figure 3.1: Parameter plane of f_c.

As seen in the picture and discussed above, if we put aside those limbs, there is a holomorphic map between this parameter plane and Mandelbrot set. To be precise, the outer part ($\hat{\mathbb{C}} \setminus$ the central disk with radius 4) corresponds to the disk centered at -1 in Mandelbrot set, and the inside cardioid corresponds to the main cardioid of Mandelbrot set. If we just put aside the small components in between for a while, this gives us a nice way to give a parameter its combinatorial address. For instance, when we talk about the rational angle, we mean the same angle for the correspondent parameter in Mandelbrot set.

3.2 Iterated Monodromy Groups for Parameters with Rational Angles

As we discussed above, there are certain kinds of parameters which can be assigned with rational angles, namely the parameters in the main components attached to the cardioid.

what we are going to do below is first give a formula of iterated monodromy group for each parameter lying in the main components, and then give parameters with deeper combinatorial address by tuning.

3.2.1 Some computation results for $IMG(f_c)$ with parameter in the main components

For this case, the corresponding limbs in Mandelbrot set are those attached to the main cardioid. The address for the limb with period k is $1 - k$. The iterations restricted to the attracting cycle will be like rotations on a parabolic flower by a rational angle, with one petal mapped to infinity. So instead one of the generator in the formula of $\mathfrak{R}(\hat{\theta})$ will be replaced by a_∞, the product of other generators. One thing worth notice is that the maps going through infinity will be of degree 2 and any other map of degree 1, which results in the appearance of permutations in the formula of a_1 and a_∞.

Example 1. *For function $f_c(z) = 1 + c/z^2$, where $c \approx -1.0969 + 1.7016i$. The attracting cycle has length 5. Here is the dynamical plane of this map*

Here we can pick the base point to be one of the fixed points. The iterations restricted to the attracting cycle is iteration on a parabolic flower, with one petal mapped into a neighborhood of ∞, which is equivalent to this following graph. For details of parabolic flower, see [5].

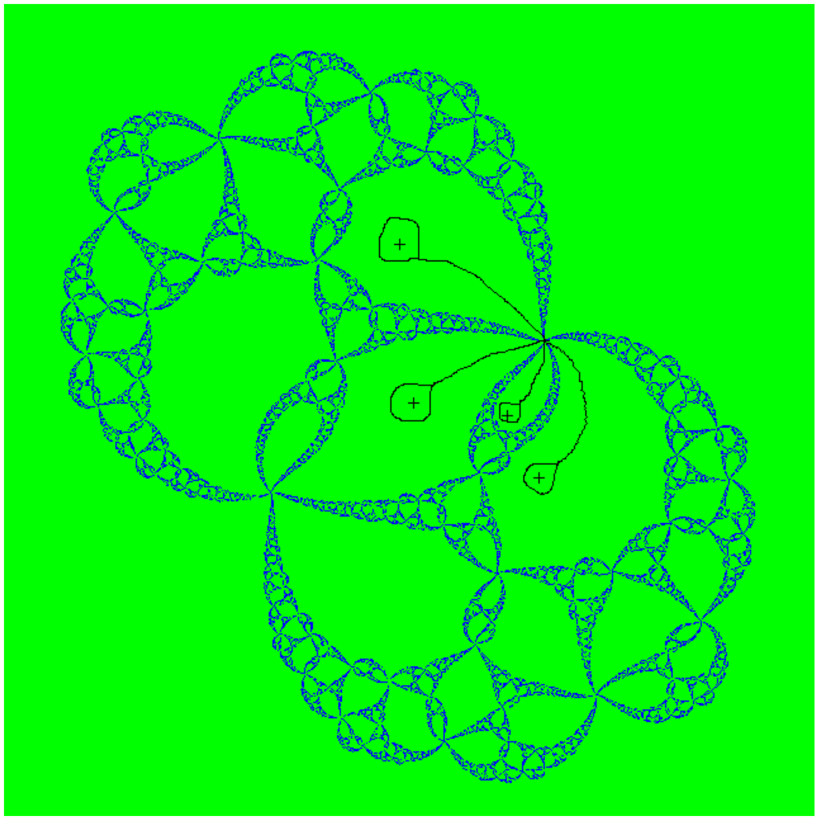

Figure 3.2: Dynamical plane of f_c with period 5.

Computing the iterated monodromy group via the Proposition 1 with the generators chosen above, we get the group

$$G = < a_1, a_2, a_3, a_4, a_\infty >$$

$$a_1 = \sigma \ll a_3^{-1}a_1^{-1}, a_4^{-1}a_2^{-1} \gg$$

$$a_2 = \ll a_1, 1 \gg$$

$$a_3 = \ll a_2, 1 \gg$$

$$a_4 = \ll a_3, 1 \gg$$

$$a_\infty = a_3^{-1}a_1^{-1}a_4^{-1}a_2^{-1} = \sigma \ll a_4, 1 \gg$$

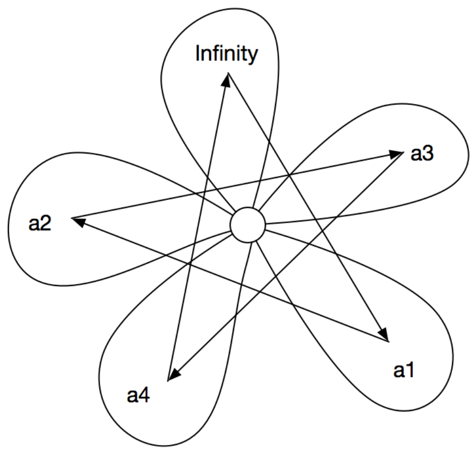

Figure 3.3: Iterations on one parabolic flower.

The pattern for such group is rather clear, and we have one more example here.

Example 2. *$c \approx -0.1838 + 2.6408i$, the attracting cycle has length 7. The Julia set and iterated monodromy group are shown as following.*

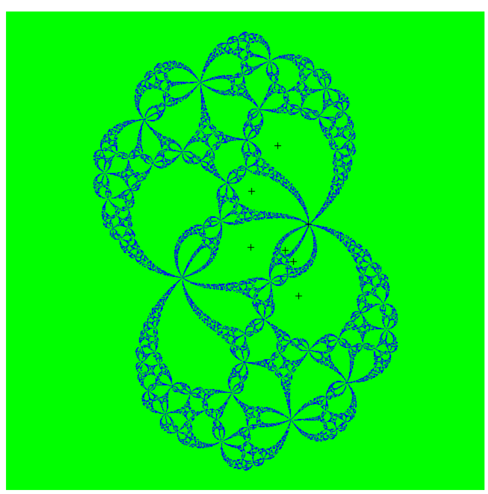

Figure 3.4: Period 7 case.

$$G =< a_1, a_2, a_3, a_4, a_5, a_6, a_\infty >$$

$$a_1 = \sigma \ll a_5^{-1} a_3^{-1} a_1^{-1}, a_6^{-1} a_4^{-1} a_2^{-1} \gg$$

$$a_2 =\ll a_1, 1 \gg$$

$$a_3 =\ll a_2, 1 \gg$$

$$a_4 =\ll a_3, 1 \gg$$

$$a_5 =\ll a_4, 1 \gg$$

$$a_6 =\ll a_5, 1 \gg$$

$$a_\infty = a_5^{-1} a_3^{-1} a_1^{-1} a_6^{-1} a_4^{-1} a_2^{-1} = \sigma \ll a_6, 1 \gg$$

Example 3. *Note in this case, the parameter* c *lies on the x-axis. Here let* $c \approx -0.2167$*, we have the following Julia set and diagram for iteration.*

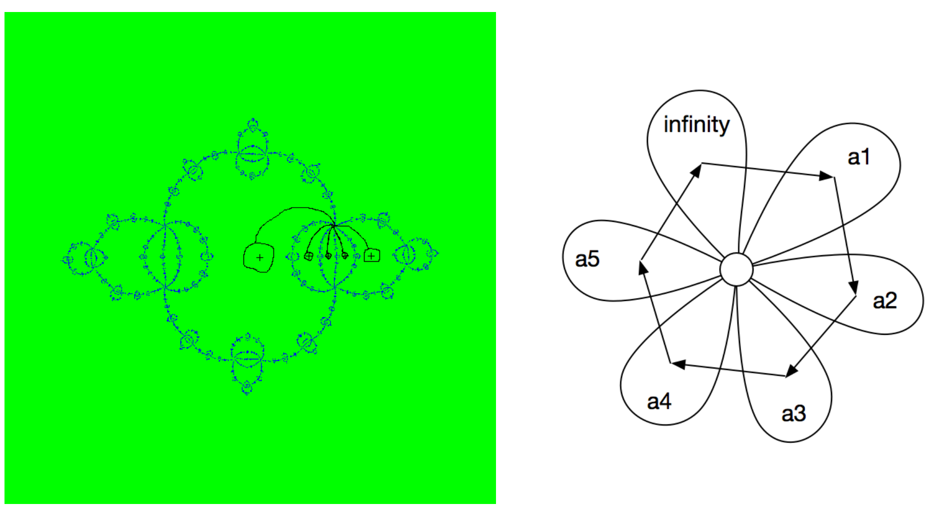

Figure 3.5: Julia set and iterations on the parabolic flower.

Computing the iterated monodromy group via the Proposition 1 with the generators chosen above, we get the group

$$G =< a_1, a_2, a_3, a_4, a_5, a_\infty >$$

$$a_1 = \sigma \ll a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}, a_5^{-1} \gg$$

$$a_2 =\ll a_1, 1 \gg$$

$$a_3 =\ll a_2, 1 \gg$$

$$a_4 =\ll a_3, 1 \gg$$

$$a_5 =\ll a_4, 1 \gg$$

$$a_\infty = a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1}a_5^{-1} = \sigma \ll a_5, 1 \gg$$

3.2.2 Some computation results for the tuning of first level

In this case, the corresponding limbs are those attached to the outer circle. The address for the parameters with period k is $1-2-k$, and the dynamics behave as jumping between two separate copies of Julia set of some quadratic polynomial. The only difference is that one component is mapped to infinity, so one of the generators in the formula of $\mathfrak{R}(\hat{\theta})$ will be replaced by a_∞. In order to get good enough expressions for later terms, the first generator a_1 will not be simply $\sigma \ll 1, a_\infty \gg$ but a conjugate of it.

Example 4. *For function $f_c(z) = 1 + c/z^2$, where $c \approx -1.786 + 3.080i$. The attracting cycle has length 6. Here is the dynamical plane of this map*

Here we choose the base point to be a fixed point and generators are picked as in the picture. Note there is another point in the attracting cycle namely ∞. By restricting the iteration in the Julia set to the attracting cycle, we have the following equivalent map which is the iteration between two parabolic flowers.

Computing the iterated monodromy group via the Proposition 1 with the generators chosen

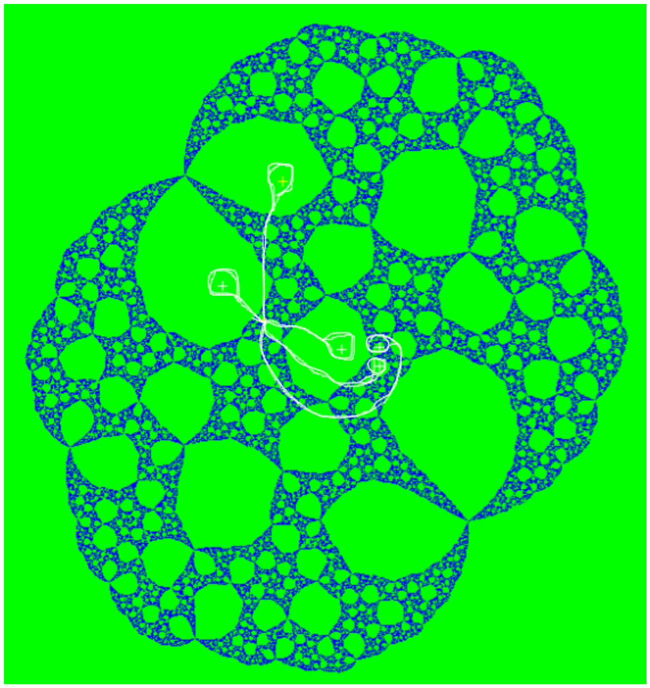

Figure 3.6: Dynamical plane of f_c with period 6.

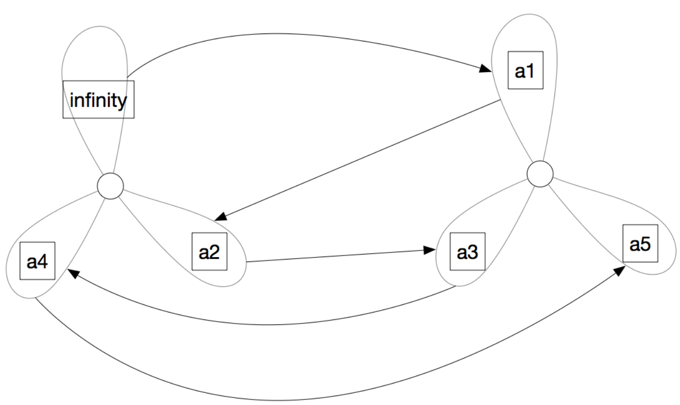

Figure 3.7: Iterations between two parabolic flowers.

above, we get the group

$$\begin{aligned}
G &=< a_1, a_2, a_3, a_4, a_5, a_\infty > \\
a_1 &= \sigma \ll [a_2 a_4]^{-1}, [a_5 a_3 a_1]^{-1} \gg \\
a_2 &=\ll a_1, 1 \gg \\
a_3 &=\ll 1, a_2 \gg \\
a_4 &=\ll a_3, 1 \gg \\
a_5 &=\ll 1, a_4 \gg \\
a_\infty &= [a_2 a_4 a_5 a_3 a_1]^{-1} = \sigma \ll a_5, 1 \gg
\end{aligned}$$

There are several other examples in this case whose iterated monodromy groups have been computed.

Example 5. *1. $c = -3$, the attracting cycle has length 4. The Julia set and iterated monodromy group are as following.*

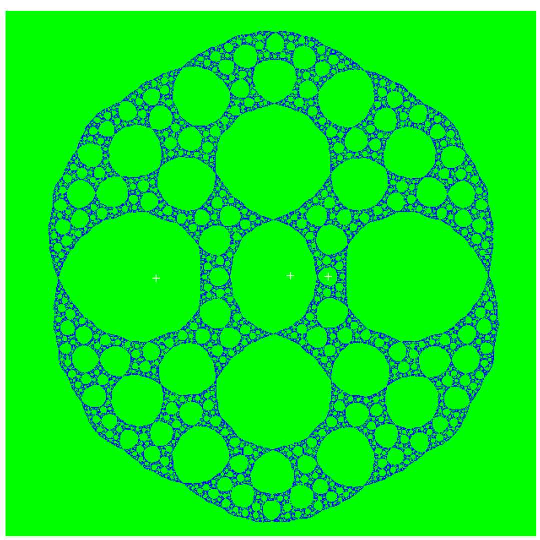

Figure 3.8: Period 4 case.

$$G =< a_1, a_2, a_3, a_\infty >$$

$$a_1 = \sigma \ll [a_2]^{-1}, [a_3 a_1]^{-1} \gg$$

$$a_2 =\ll a_1, 1 \gg$$

$$a_3 =\ll 1, a_2 \gg$$

$$a_\infty = [a_2 a_3 a_1]^{-1} = \sigma \ll a_3, 1 \gg$$

2. *$c \approx 0.014 + 3.741i$, the attracting cycle has length 8. The Julia set and iterated monodromy group are as following.*

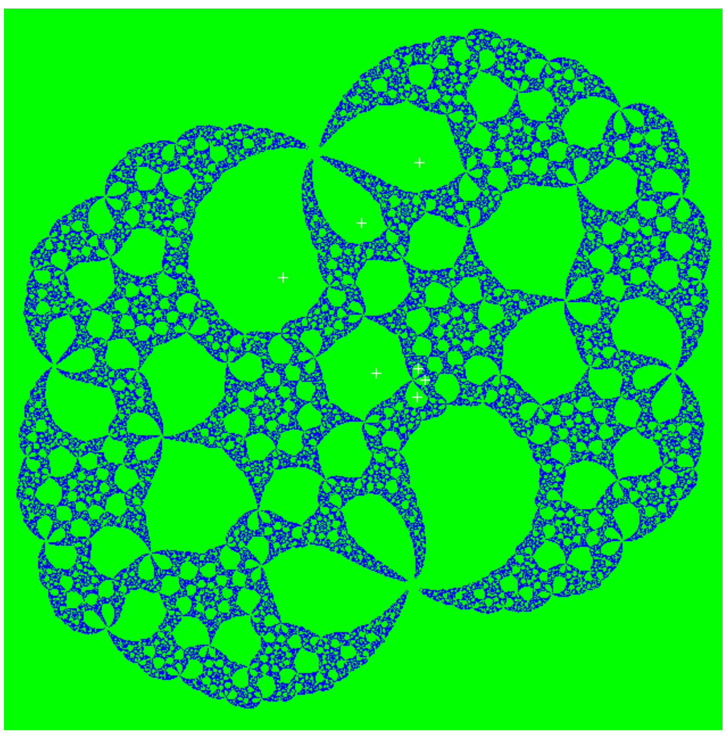

Figure 3.9: Period 8 case.

$$G = < a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_\infty >$$

$$a_1 = \sigma \ll [a_2 a_4 a_6]^{-1}, [a_7 a_5 a_3 a_1]^{-1} \gg$$

$$a_2 = \ll a_1, 1 \gg$$

$$a_3 = \ll 1, a_2 \gg$$

$$a_4 = \ll a_3, 1 \gg$$

$$a_5 = \ll 1, a_4 \gg$$

$$a_6 = \ll a_5, 1 \gg$$

$$a_7 = \ll 1, a_6 \gg$$

$$a_\infty = [a_2 a_4 a_6 a_7 a_5 a_3 a_1]^{-1} = \sigma \ll a_7, 1 \gg$$

3. *$c \approx 1.2 + 3.6i$, the attracting cycle has length 10. The Julia set and iterated monodromy group are as following.*

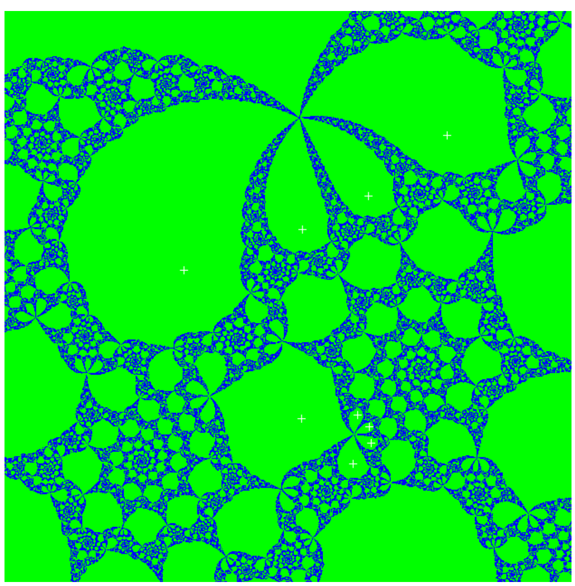

Figure 3.10: Period 10 case.

$$G = < a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_\infty >$$

$$a_1 = \sigma \ll [a_2 a_4 a_6 a_8]^{-1}, [a_9 a_7 a_5 a_3 a_1]^{-1} \gg$$

$$a_2 = \ll a_1, 1 \gg$$

$$a_3 = \ll 1, a_2 \gg$$

$$a_4 = \ll a_3, 1 \gg$$

$$a_5 = \ll 1, a_4 \gg$$

$$a_6 = \ll a_5, 1 \gg$$

$$a_7 = \ll 1, a_6 \gg$$

$$a_8 = \ll a_7, 1 \gg$$

$$a_9 = \ll 1, a_8 \gg$$

$$a_\infty = [a_2 a_4 a_6 a_8 a_9 a_7 a_5 a_3 a_1]^{-1} = \sigma \ll a_9, 1 \gg$$

Note in this case, the iterations restricted to the attracting cycles are all iterations between two parabolic flowers, with one petal mapped into a neighborhood of ∞. So basically the group may looks like the iteration monodromy groups of some quadratic polynomials, with one of the generators to be a_∞ the multiplication of other generators.

3.2.3 Formula for iteration monodromy groups in above cases

For the parameter plane of function $f_c(z)$, we have the following lemma:

Lemma 1. *Let C be the main cardioid in the parameter plane, and $B(0,4)$ be the disk centered at 0 with radius 4. Then $\mathbb{C} \setminus (B(0,4) \setminus C)$ is bi-holomorphically isomorphic to the main component of Mandelbrot set (set of parameters with period 1 or 2). Specially, on the boundary we can assign a rational angle for certain points.*

Proof. Consider parameter $c \in C$, then f_c has an attracting fixed point α_c for each parameter c.

Since α_c is an attracting fixed point, $|f'(\alpha_c)| < 1$, in other words $|f'(\alpha_c)| \in D$ where D is the unit disk. Then the map $\phi : c \mapsto f'(\alpha_c)$ gives a bi-holomorphic isomorphism between C and unit disk. In the same way we can define a bi-holomorphic isomorphism ψ from the main cardioid of Mandelbrot set to unit disk. Then the map $\psi^{-1} \circ \phi$ gives the bi-holomorphic isomorphism between C and main cardioid of Mandelbrot set, which combined with the map between disk centered at -1 and disk centered at ∞ gives the bi-holomorphic isomorphism between $\mathbb{C} \setminus (B(0,4) \setminus C)$ and the main component of Mandelbrot set.

The map $\phi : c \mapsto f'(\alpha_c)$ can be extended to the boundary. For those parameter c on the boundary such that $f'(\alpha_c)$ is a rational number, we assign this rational number as a rational angle to the parameter. On the other hand, this rational angle is exactly the rotation angle of iterations on the Julia set. $\square$

Theorem 1. *For which case we discussed above, any parameter c can be signed with a rational angle $\hat{\theta}(c)$ as proved in Lemma 1. Assume $\hat{\theta}(c) = r/k$, where r and k are positive integers, then the iterated monodromy group of $f_c(z)$ has the following form*

$$G =< a_1, a_2, a_3, \ldots, a_{k-1}, a_\infty >$$

$$a_1 = \sigma \ll a^{-1}_{[m]} a^{-1}_{[2m]} \ldots a^{-1}_{[(k-r-1)m]}, a^{-1}_{[k-1]} a^{-1}_{[m-1]} a^{-1}_{[2m-1]} \ldots a^{-1}_{[(r-1)m-1]} \gg$$

$$a_{i+1} =\ll a_i, 1 \gg$$

$$a_\infty = a^{-1}_{[m]} a^{-1}_{[2m]} \ldots a^{-1}_{[(k-r-1)m]} a^{-1}_{[k-1]} a^{-1}_{[m-1]} a^{-1}_{[2m-1]} \ldots a^{-1}_{[(r-1)m-1]} = \sigma \ll a_{k-1}, 1 \gg$$

Where $1 \leq i \leq k-2$, $[j]$ is the smallest positive integer in congruence class of j modulo k, and m is the inverse of r modulo k.

And for parameters shown in the last examples, the iterated monodromy groups are the tunings of group G_0 by those iterated monodromy groups given above, where $G_0 =< a >$, $a = \sigma \ll a^{-1}, 1 \gg$.

Proof. Such group description comes from our certain choice of generators. For parameter c with angle $\hat{\theta}(c) = r/k$, the attraction cycle has length k, which is $\infty \Rightarrow \alpha_1 \to \alpha_2 \to \cdots \to \alpha_{k-1} \Rightarrow \infty$. Then we choose the base point to be the fixed point inside this cycle, and generator a_i to be the loop around α_i. By Lemma 1, this angle $\hat{\theta}(c) = r/k$ is the rotation angle on the parabolic flower, so the clockwise order of these generators will be

$$a_{[m]}, a_{[2m]}, \ldots, a_{[(k-r-1)m]}, a_{[k-1]}, a_{[m-1]}, a_{[2m-1]}, \ldots, a_{[(r-1)m-1]}$$

So if we choose the connecting path between preimages of base point to be the preimage of a_∞, we will have the iterated monodromy group described in the way above. For more detail, see example 1 and 3. □

3.3 Iterated Monodromy Groups for Parameters with Deeper Combinatorial Address

Generally speaking, when we cross a boundary of some component in parameter plane and go deeper, what happens is that each component of the Julia set is tuned by some fractals. The iterated monodromy groups will be the abstract tuning of the former group with another iterated monodromy group. See the following picture which is an example of period 5 case.

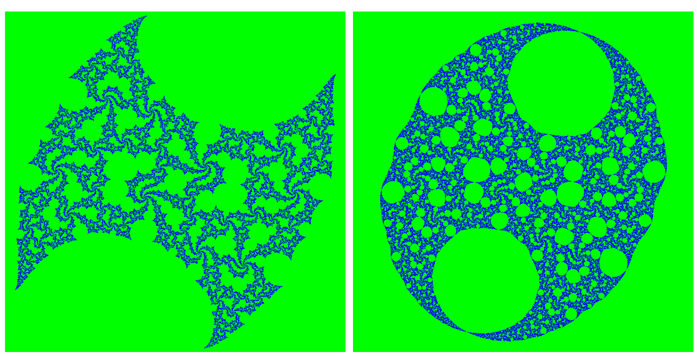

Figure 3.11: An example of tuning.

In this case the new iterated monodromy group we get is a tuning of $G(2/5)$. We can define the combinatorial address with form $1 - * - * - \cdots - *$ equivalent to the rational angle for a parameter c, which uniquely determines the iterated monodromy group for this parameter by this tuning method. Such combinatorial address can be assigned to any parameter lying inside the limb which can be connected to the main cardioid. The combinatorial address is given in the following way:

We start from the main cardioid. Any parameter in the main cardioid has combinatorial address 1. Then for any parameter in the limb attached to the main cardioid with period k_1 (the attraction cycle has length k_1), we give them address $1 - k_1$. Then we define inductively, for any parameter in the limb with period k_n attached to limb with combinatorial address $1 - k_1 - k_2 - \cdots - k_{n-1}$, we give them combinatorial address $1 - k_1 - k_2 - \cdots - k_{n-1} - k_n$. Then the parameter with such combinatorial address satisfies the following property.

Proposition 2. *The iterated monodromy group of parameter with address* $1 - k_1 - k_2 - \cdots - k_\ell$ *is the tuning of iterated monodromy group of parameter with address* $1 - k_1 - k_2 - \cdots - k_{\ell-1}$.

4. SUMMARY AND CONCLUSIONS

In this thesis, we give a description of iterated monodromy groups for a large class of parameters of function $f_c(z) = 1 + c/z^2$, namely all parameters which can be connected to the main cardioid. We first give a formula for iterated monodromy groups with parameters in the main cardioid and then get those groups with deeper combinatorial address via the tuning method.

4.1 Challenges

The challenge here is that though the tuning method is quite straight forward to understand, it is not easy to give uniform formulas for an arbitrary tuning. So the explicit formula for parameters with deeper address will depend. We cannot write down the formulas, that's the reason why we give them via this method.

Another challenge is to determine the exact center of each component in the parameter plane. Although we know that all iterated monodromy groups are isomorphic for parameters in the same component, it will be nice if we can find the explicit coordinate of the center. However, to find the center we need some algorithm, for example the spider algorithm for Mandelbrot set, due to the limited time we don't give the coordinates of those centers in this thesis, we just use approximate coordinates instead.

4.2 Further Study

For our function $f_c(z) = 1 + c/z^2$, there are still some other parameters in the parameter plane which can not be easily signed a combinatorial address. For these parameters, maybe we need some new approach and that is what should be studied next.

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